$\mathbb{C}$: • je těleso, algebraicky uzávěr $\mathbb{R}$:

• alg. uz.: a) alg. rozšíření $\mathbb{R}$

všechny prvky $\mathbb{C}$ jsou alg. nad $\mathbb{R}$

(tj. jsou kořeny něj. polynomu nad $\mathbb{R}$)

b) algebraicky uzavřené těleso, tj.

libovolný polynom $P \in \mathbb{C}[x]$, $st(P) \geq 1$

aspoň jeden kořen v $\mathbb{C}$.

Definice: Komplexní číslo je prvek $\mathbb{R}^2 =: \mathbb{C}$.

na $\mathbb{C}$ def. operace: • $(a,b) = (c,d) \iff a=c \wedge b=d$

• $(a,b) + (c,d) = (a+c, b+d)$

• $(a,b) \cdot (c,d) = (ac - bd, ad + bc)$

Tvrzení: $(\mathbb{C}, +, -, \cdot, ^{-1}, (0,0), (1,0))$ je těleso.

• „$\mathbb{R} \subseteq \mathbb{C}$": Uvažujeme $f: \mathbb{R} \to \mathbb{C}$

$f(r) = (r,0)$. Pak:

• $f(r+s) = (r+s, 0) = (r,0) + (s,0) = f(r) + f(s)$

• $f(r \cdot s) = (r \cdot s, 0) = (r \cdot s - 0 \cdot 0, r \cdot 0 + 0 \cdot s) =$

$= (r,0) \cdot (s,0) = f(r) \cdot f(s)$.

• $f(1) = (1,0)$ • $f(0) = (0,0)$.

$f$ je triviálně prosté endomorfismus $\mathbb{R}$ do $\mathbb{C}$.

Ztotožníme $f(\mathbb{R})$ s $\mathbb{R}$.

$\mathbb{R} \approx f(\mathbb{R}) = \{(r,0) : r \in \mathbb{R}\} \subseteq \mathbb{R}^2 = \mathbb{C}$.

• místo $f(r) (= (r,0))$ píšeme $r$.

„$\mathbb{C} = \mathbb{R} + i\mathbb{R}$": Píšeme $i := (0,1)$. Pak

$\mathbb{C} \ni (a,b) = (a,0) + (0,b) = (a,0) + (b,0) \cdot (0,1) =$

$= a + b \cdot i$, $a, b \in \mathbb{R}$.

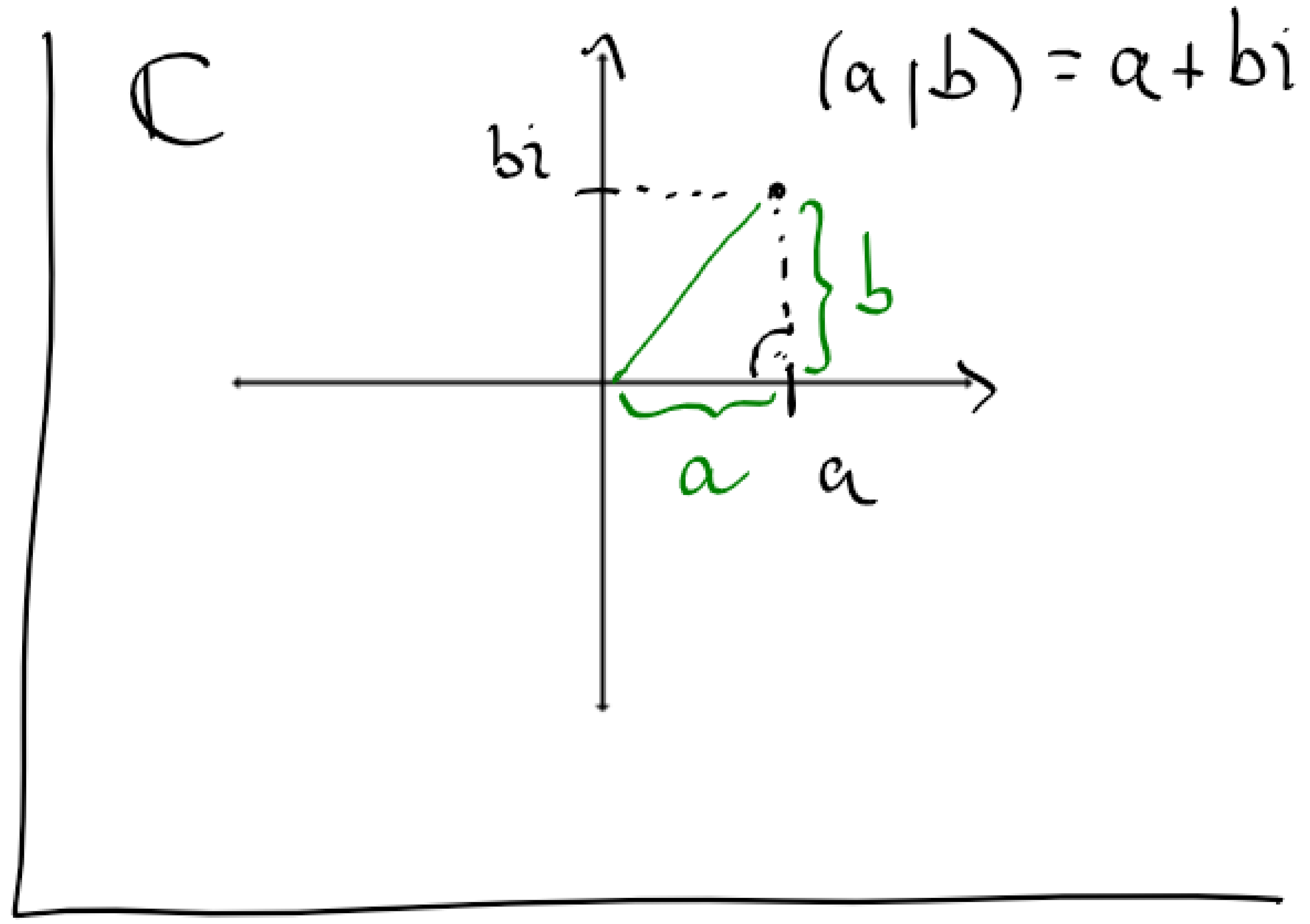
**Značení:** • $\overline{a+bi} = a-bi$

$(z = a+bi$, pak $\overline{z} = a-bi)$

• $|a+bi| = \sqrt{a^2+b^2}$ ... modulus.

• $\mathrm{Re}(a+bi) = \mathrm{Re}((a,b)) = a$

• $\mathrm{Im}(a+bi) = \mathrm{Im}(a,b) = b$




**Pozorování:**

• $i^2 = -1$ $\quad [\,i^2 = (0,1)\cdot(0,1) = (-1,0) = -1 \;\boxtimes\,]$

• $(a+bi)\overline{(a+bi)} = a^2+b^2$

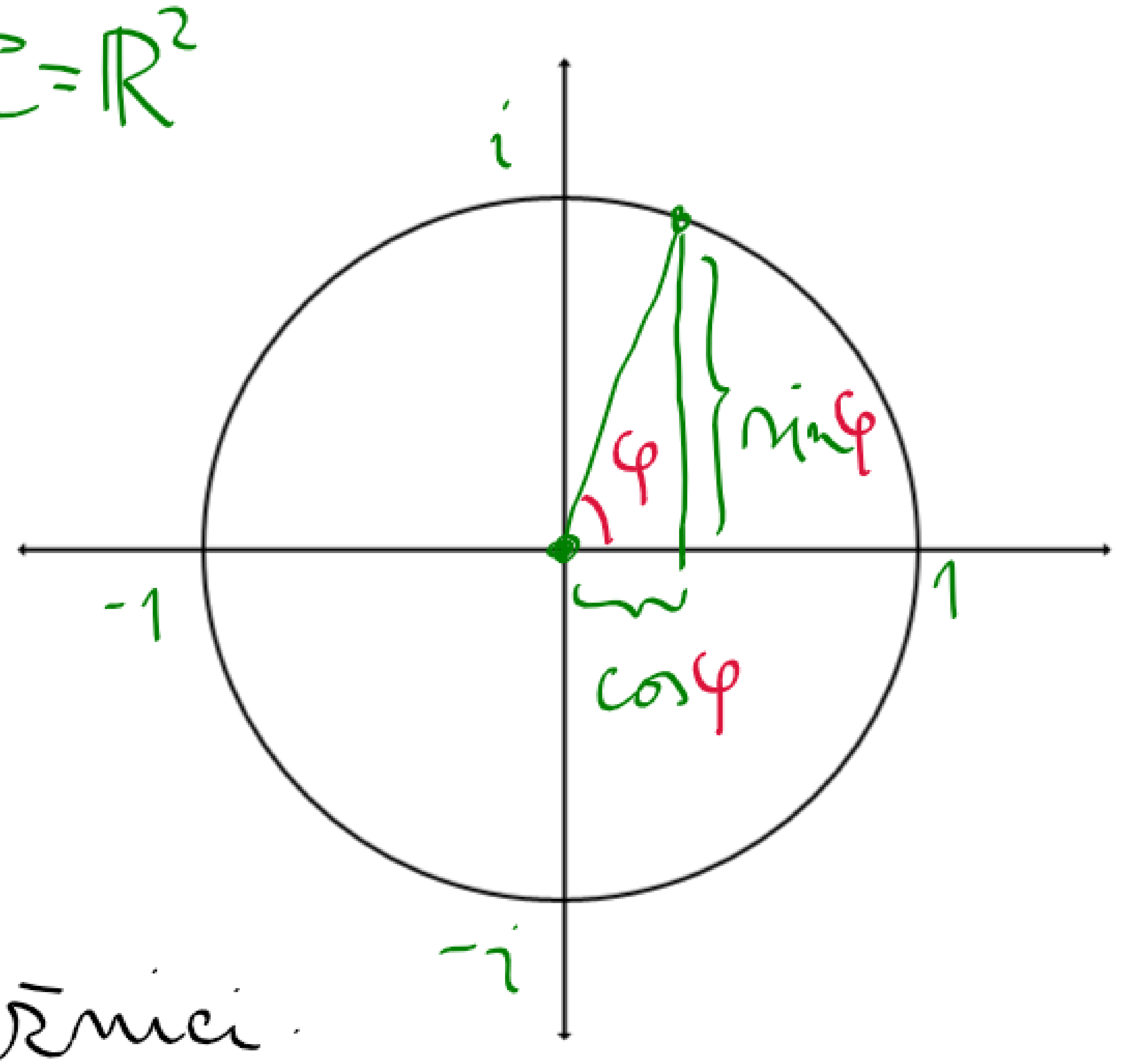
$[\,(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 - b^2(-1). \;\boxtimes\,]$

• Tedy $\boxed{z\cdot\overline{z} = |z|^2}$ $\quad$ • $z = a+bi = 0 \iff a=0 \wedge b=0$

• Pro $z \neq 0$: $\; z\cdot\frac{\overline{z}}{|z|^2} = 1$, tj. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

**Příklad:** • $z = 3+4i$ $\qquad \frac{1}{z} = \frac{3-4i}{25}$

**Goniometrický tvar:** $\mathbb{C} = \mathbb{R}^2$




1) Položme-li

$k(\varphi) = (\cos\varphi, \sin\varphi)$

$\varphi \in (-\pi, \pi]$, pak

$k(\varphi)$ probíhá 1-kružnici.

2) $r$-kružnice $(r>0)$

$r\cdot k(\varphi) = \boxed{(r\cdot\cos\varphi, r\cdot\sin\varphi)}$ je param.

$r$-kružnice.

3) $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C}\setminus\{0\}$ libovolný bod,

tak $(x,y)$ leží na $r$-kružnici, kde $r = \sqrt{x^2+y^2}$

Tedy pro něj $\varphi \in (-\pi, \pi]$ platí

$(x,y) = (r\cdot \cos\varphi \,,\, r\cdot \sin\varphi)$ $\quad (r>0)$

3*) v řeči $\mathbb{C}$: $\forall z \in \mathbb{C}\setminus\{0\}$ $\exists!\, r>0$

$\exists!\, \varphi \in (-\pi,\pi]$: $\quad z = r\cdot\cos\varphi + i\cdot r\cdot\sin\varphi =$

$= r\cdot(\cos\varphi + i\cdot\sin\varphi)$

Tedy komplexní číslo $z \in \mathbb{C}\setminus\{0\}$ určují velikost $(r)$ a úhel $(\varphi)$

Úhel $\varphi$ je určen jednoznačně v lib. intervalu (polouz.) délky $2\pi$.

Značíme $\operatorname{Arg}(z) := \varphi \in (-\pi, \pi]$

$\arg(z) := \{\operatorname{Arg}(z) + 2k\pi : k \in \mathbb{Z}\}$

## Určování souřadnic (kartézské, polární)

Příklad: $z = \left(\frac{1+i}{1-i}\right)^3 \qquad \operatorname{Re} z = ? \quad \operatorname{Im} z = ?$

Víme $\alpha \in \mathbb{C}$: $\quad \alpha\cdot\bar{\alpha} = |\alpha|^2 \in \mathbb{R}$ !

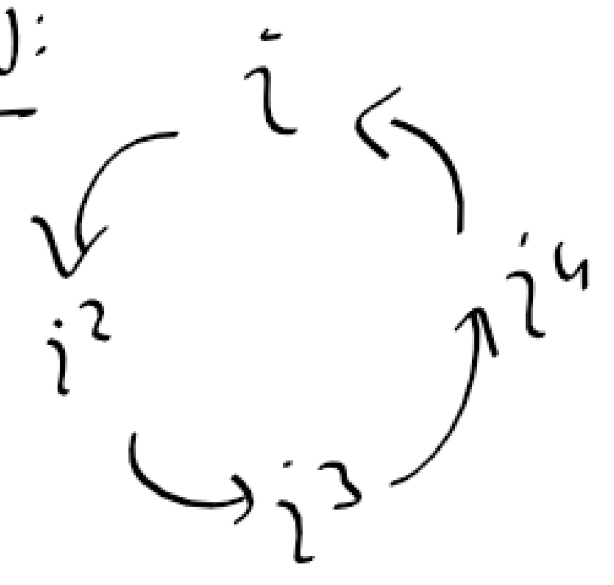
$$z = \frac{(1+i)^3}{(1-i)^3}\cdot\frac{(1+i)^3}{(1+i)^3} = \frac{(1+i)^6}{(1-i^2)^3} =$$

$$= \frac{(1+i)^6}{2^3} = \frac{1}{8}\left(1+6i+15i^2+20i^3+15i^4+6i^5+i^6\right)$$

$$= \frac{1}{8}\left(\cancel{1}+6i-\cancel{15}-20i+\cancel{15}+6i-\cancel{1}\right)$$

$$= \frac{1}{8}(-8i) = -i$$

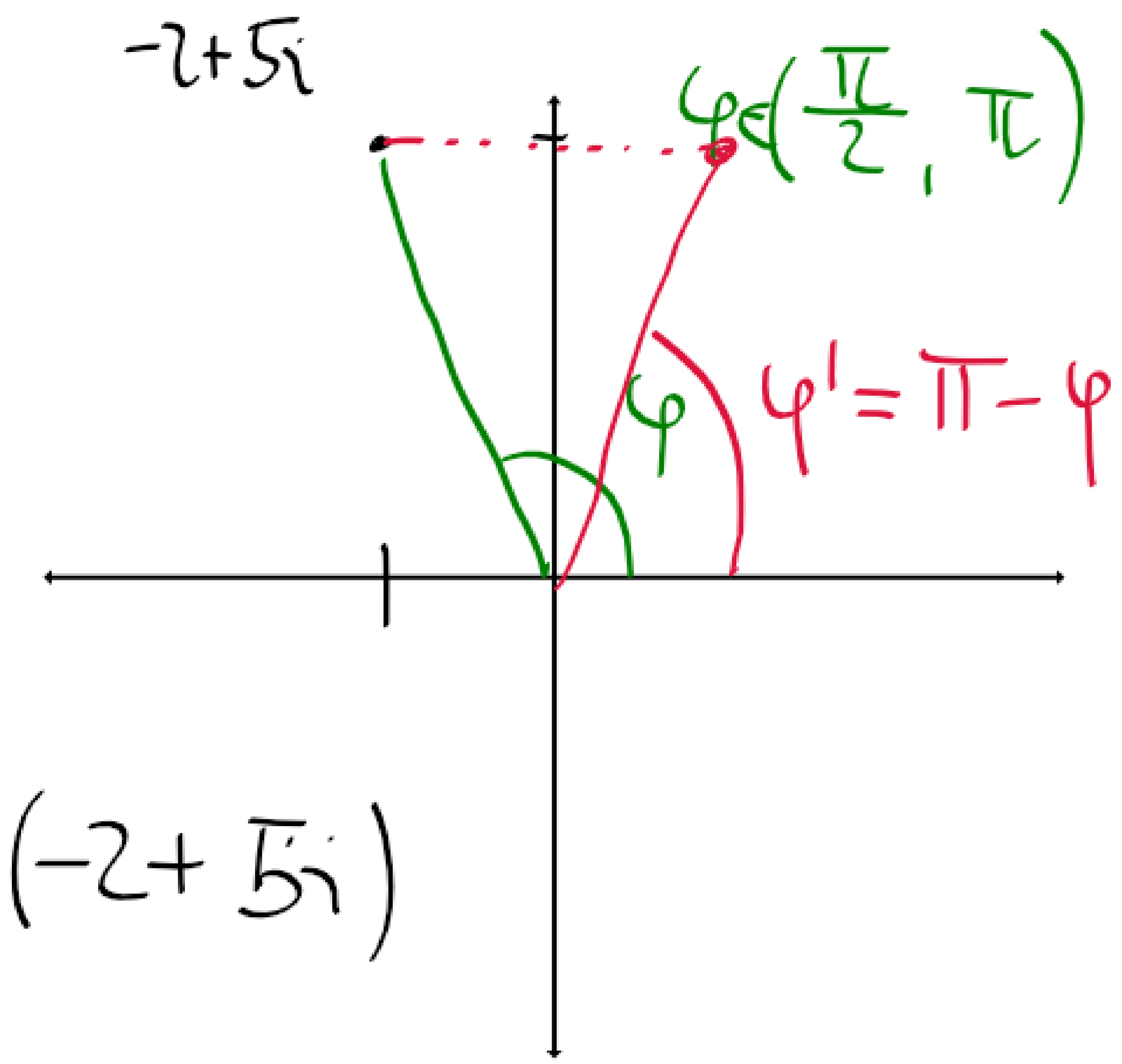
POZN:




$$\alpha = \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i+i^2}{2} =$$

$$= \frac{2i}{2} = i \quad \text{Tedy } \alpha^3 = i^3 = -i$$

**Příkl.:** $\mathrm{Arg}(-2+5i)$




$\operatorname{tg}\varphi' = \frac{5}{2} \qquad \varphi' \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\varphi' = \operatorname{arctg}\frac{5}{2}$

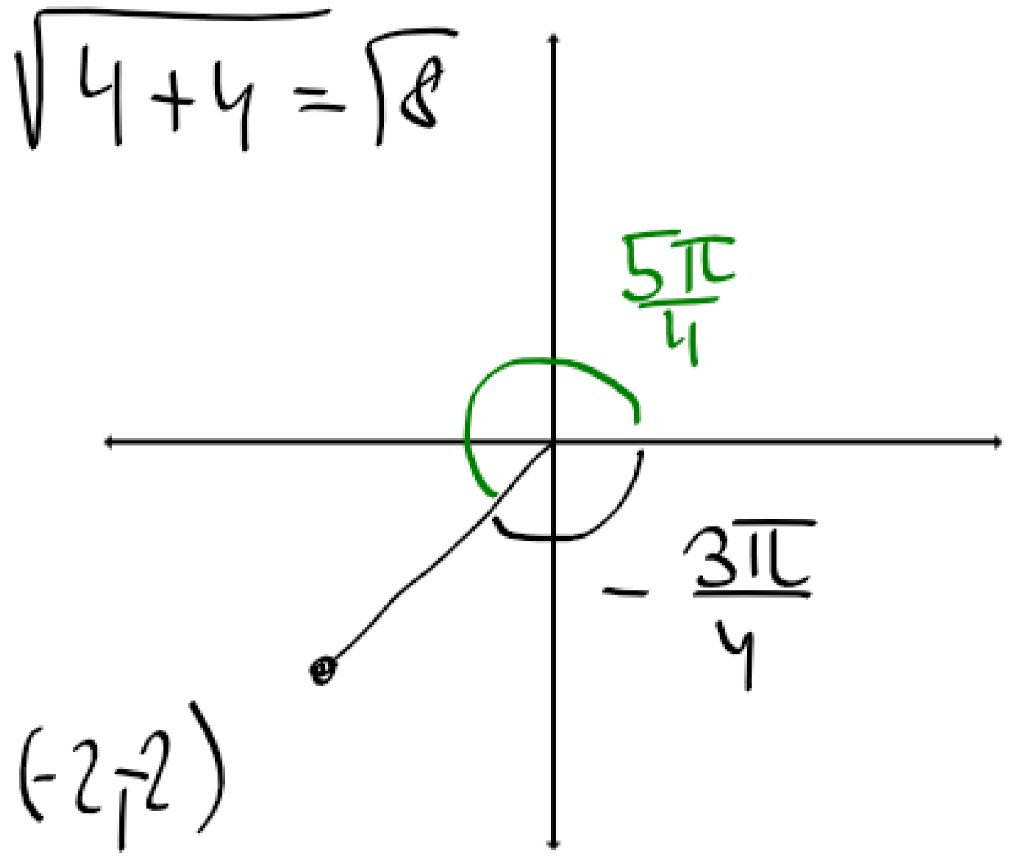
$\varphi = \pi - \operatorname{arctg}\frac{5}{2} = \mathrm{Arg}(-2+5i)$

$|-2+5i| = \sqrt{4+25} = \sqrt{29}$

$-2+5i = \sqrt{29}\cdot\left(\cos\left(\pi - \operatorname{arctg}\frac{5}{2}\right) + i\sin\left(\pi - \operatorname{arctg}\frac{5}{2}\right)\right)$

**Příklad:** $z = -2-2i$ $\quad\Big|\quad |z| = \sqrt{4+4} = \sqrt{8}$

$z = \sqrt{8}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$



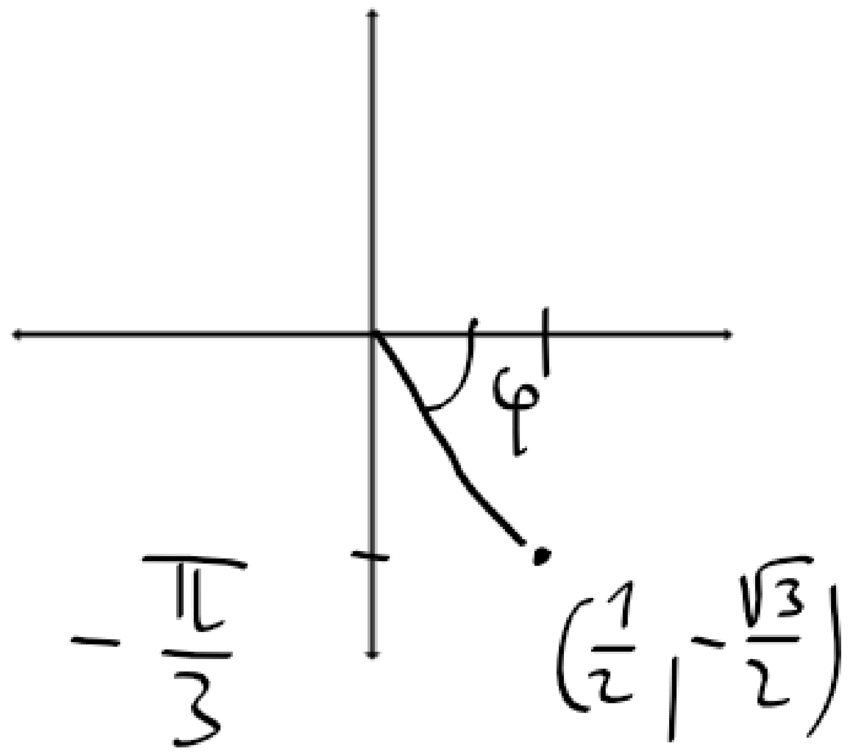

**Zk:** $\ldots = -2-2i$

**Příklad:** • $1 + i^{123} = 1 + i^{123 \bmod 4} = 1 + i^3 = 1 - i$

**Příklad:** $\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3$.

$z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\operatorname{tg}\varphi = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

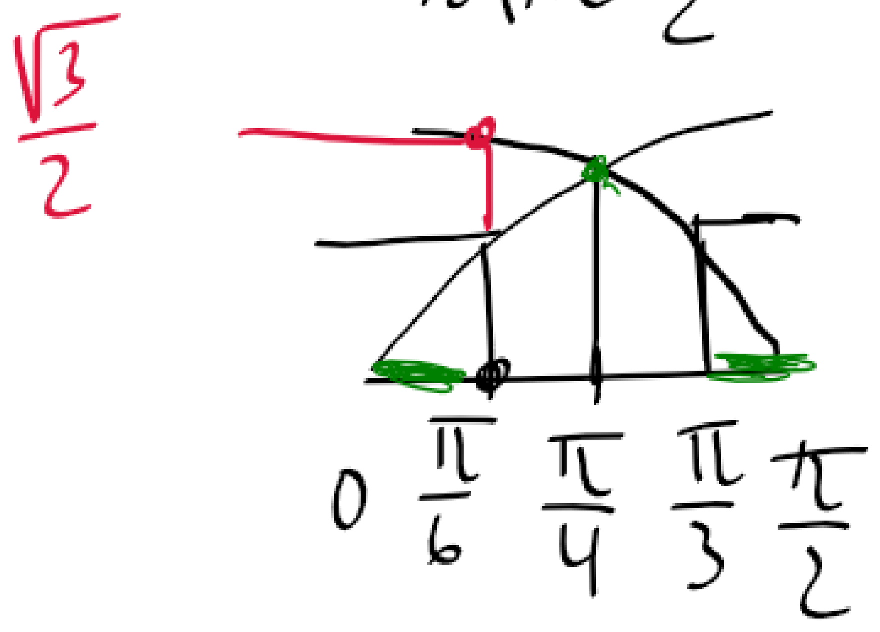



$\varphi = \operatorname{arctg}(-\sqrt{3}) = -\operatorname{arctg}\sqrt{3} = -\frac{\pi}{3}$

**POZN.** Trik: $\frac{\pi}{6} \cong \frac{3}{6} = \frac{1}{2}$

$\sin\frac{1}{2} \approx \frac{1}{2} \qquad \sin\frac{\pi}{6} = \frac{1}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$.




$|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

$z = 1\cdot\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = \cos\left(\frac{\pi}{3}\right) - i\sin\frac{\pi}{3}$

$z^3 = \cos\left(3\cdot\frac{\pi}{3}\right) + i\sin\left(3\cdot\frac{\pi}{3}\right) = -1 + i0 = \underline{\underline{-1}}$

MOIVREOVA VĚTA: $z = r\cdot(\cos\varphi + i\sin\varphi)$, $n \in \mathbb{N}$

Pak: $z^n = r^n \cdot \left(\cos(n\cdot\varphi) + i\sin(n\varphi)\right)$.

Důkaz: Pozorování:

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) =$$

$$= (\cos\alpha\cdot\cos\beta - \sin\alpha\sin\beta) +$$

$$(\cos\alpha\cdot i\sin\beta + i\sin\alpha\cdot\cos\beta) =$$

$$= \cos\alpha\cdot\cos\beta - \sin\alpha\sin\beta +$$

$$i\cdot(\sin\alpha\cos\beta + \cos\alpha\sin\beta) =$$

$$= \cos(\alpha+\beta) + i\cdot\sin(\alpha+\beta).$$

$$z^n = r^n\cdot(\cos\varphi + i\sin\varphi)^n =$$

$$= r^n\cdot(\cos\varphi + i\sin\varphi)\cdot(\cos\varphi + i\sin\varphi)\cdot\ldots\cdot(\cos\varphi + i\sin\varphi)$$

$$= r^n\cdot(\cos 2\varphi + i\sin 2\varphi)\cdot\ldots\cdot(\cos\varphi + i\sin\varphi)$$

$$= r^n\cdot\cos n\varphi + i\sin n\varphi. \quad \boxtimes$$

Příklad: Řešte v $\mathbb{C}$ rovnici $z^3 = 1 =$

$$\boxed{z = \sqrt[3]{1}}$$

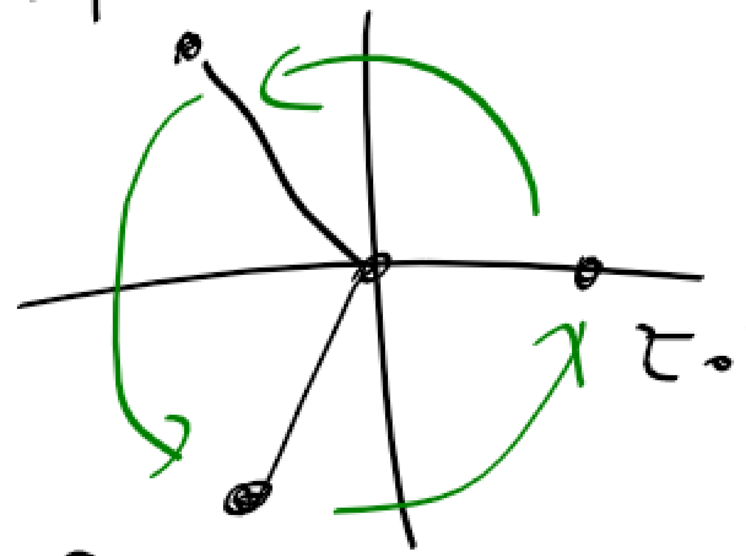
$$z = r\cdot(\cos\varphi + i\sin\varphi)$$

$$z^3 = r^3(\cos 3\varphi + i\sin 3\varphi) = 1 = 1\cdot(\cos \overset{2k\pi}{0} + i\sin \overset{2k\pi}{0})$$

Tedy: $r^3 = 1$, $r \in \mathbb{R}$, tedy $r = 1$.

- $3\varphi = 2k\pi$, $k \in \mathbb{Z}$

$$\varphi = \frac{2k\pi}{3} \in (-\pi, \pi]$$




| | |
|---|---|
| $\varphi_0 = 0$ | $z_0 = 1$ |
| $\varphi_1 = \frac{2\pi}{3}$ | $z_1 = 1\cdot\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ |
| $\varphi_{-1} = \frac{-2\pi}{3}$ | $z_{-1} = 1\cdot\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right),$ |

Př.: Popište

$$\{ z \in \mathbb{C} : |z-i| + |z+i| < 4 \} .$$